On Perron-Frobenius property of matrices having some negative entries

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Abstract

We extend the theory of nonnegative matrices to the matrices that have some negative entries. We present and prove some properties which give us information, when a matrix possesses a Perron-Frobenius eigenpair. We apply also this theory by proposing the Perron-Frobenius splitting for the solution of the linear system $Ax = b$ by classical iterative methods. Perron-Frobenius splittings constitute an extension of the well known regular splittings, weak regular splittings and nonnegative splittings. Convergence and comparison properties are given and proved.

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1 Introduction

In 1907, Perron [14] proved that the dominant eigenvalue of a matrix with positive entries is positive and the corresponding eigenvector could be chosen to be positive. With the term *dominant eigenvalue* we mean the eigenvalue which corresponds to the spectral radius. Later in 1912, Frobenius [7] extended this result to irreducible nonnegative matrices. Since then the well known *Perron-Frobenius* theory has been developed, for nonnegative matrices and the well known *Regular, Weak Regular* and *Nonnegative Splittings* have been introduced and developed for the solution of large sparse linear systems by iterative methods (Varga [16], Young [20], Berman and Plemmons [2], Bellman [1], Woźnicki [18], Csordas and Varga [5], Neumann and Plemmons [10], Miller and Neumann [9], Marek and Szyld [8], Woźnicki [19], Climent and Perea [4]). (An excellent account of all sorts of splittings can be found in Nteirmentzidis [12]). Such linear systems are yielded from the discretisation of elliptic and parabolic partial differential equations, from integral equations, from Markov chains and from other applications (see, e.g., [2]). In 1985, O'Leary and White [13] introduced the theory of Multisplittings which is very useful for the solution of linear systems on parallel computer architectures. Since then many researchers, based on their theory, have proposed various Multisplitting techniques (Neumann and Plemmons [11], Bru, Elsner and Neumann [3], Elsner [6], White [17] and others).

Recently, Tarazaga, Raydan and Hurman [15], have given a sufficient condition that guarantees the existence of the Perron-Frobenius eigenpair, for the class of symmetric matrices

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which have some negative entries. Their result was obtained by studying some convex and closed cones of matrices.

It is obvious, from the continuity of the eigenvalues and the entries of the eigenvectors, as functions of the entries of matrices, that the Perron-Frobenius theory may hold also in the case where the matrix has some absolutely small negative entries. This observation brings up some questions. E.g., How small could these entries be? What is their distribution? When such a matrix looses the Perron-Frobenius property? These questions are very difficult to answer. Tarazaga et al in [15] gave a partial answer to the first question by providing a sufficient condition for the symmetric matrix case.

In this paper the behavior of such matrices is studied. Sufficient and necessary conditions as well as monotonicity properties are stated and proved, for the general case of real matrices. So, we answer implicitly the above questions by extending the Perron-Frobenius theory of nonnegative matrices to the class of matrices that possess the Perron-Frobenius property. Finally, we apply this theory by introducing the *Perron-Frobenius splitting* for the solution of linear systems by classical iterative methods. This splitting is an extension and a generalization of the well known regular, weak regular and nonnegative splittings. We also present and prove convergence and comparison properties for the proposed splitting.

This work is organized as follows: In Section 2 the main results of the extension of the Perron-Frobenius theory are stated and proved. In Section 3 we propose the Perron-Frobenius splitting and give convergence and comparison properties based on it. As the theory is being developed, various numerical examples are given in the text to illustrate it.

2 Extension of the Perron-Frobenius theory

We begin with our theory by giving two definitions:

Definition 2.1 *A matrix $A \in \mathbb{R}^{n,n}$ possesses the Perron-Frobenius property if its dominant eigenvalue λ_1 is positive and the corresponding eigenvector $x^{(1)}$ is nonnegative.*

Definition 2.2 *A matrix $A \in \mathbb{R}^{n,n}$ possesses the strong Perron-Frobenius property if its dominant eigenvalue λ_1 is positive, simple ($\lambda_1 > |\lambda_i|$, $i = 2, 3, \cdots, n$) and the corresponding eigenvector $x^{(1)}$ is positive.*

It is noted that Definition 2.1 is the most general of the relevant ones given so far. The analogous definition in the well known Perron-Frobenius theory is that for nonnegative matrices. On the other hand, in Definition 2.2 a subset of matrices of Definition 2.1 is defined, which is analogous to that of irreducible and primitive nonnegative matrices. The next two theorems give sufficient and necessary conditions for the second class of matrices.

Theorem 2.1 *For a symmetric matrix $A \in \mathbb{R}^{n,n}$ the following properties are equivalent:*
i) A possesses the strong Perron-Frobenius property.
ii) There exists an integer $k_0 > 0$ such that $A^k > 0 \ \forall \ k \geq k_0$.

Proof: $(i \Rightarrow ii)$: Since A possesses the strong Perron-Frobenius property, its eigenvalues can be ordered as follows:

$$\lambda_1 = \rho(A) > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|,$$

where λ_1 is a simple eigenvalue with the corresponding eigenvector $x^{(1)} \in I\!R^n$ being positive. We choose an arbitrary nonnegative vector $x^{(0)} \in I\!R^n$ with $\|x^{(0)}\|_2 = 1$. We expand $x^{(0)}$ as a linear combination of the eigenvectors of A: $x^{(0)} = \sum_{i=1}^n c_i x^{(i)}$. Since A is symmetric the eigenvectors constitute an orthogonal basis. So, the coefficients c_i's are the inner products $c_i = (x^{(0)}, x^{(i)}),\ \ i = 1, 2, \cdots, n$, which means that $c_1 > 0$. We apply now the theorem of the Power method. So, the limit of $A^k x^{(0)}$ tends to the eigenvector $x^{(1)}$ as k tends to infinity. This means that for a certain $x^{(0)} \geq 0$ there exists an m such that $A^k x^{(0)} > 0$ for all $k \geq m$. If we choose the largest of all m's over all initial choices $x^{(0)} \geq 0$, specifically

$$k_0 \ = \max_{0 \leq x^{(0)} \in I\!R^n,\ \|x^{(0)}\|_2 = 1} \left\{ m \mid Ax^k > 0 \ \forall \ k \geq m \right\},$$

we take that for all $x^{(0)} \geq 0$, $A^k x^{(0)} > 0$ for all $k \geq k_0$, which proves our assertion.

($ii \Rightarrow i$): From the Perron-Frobenius theory of nonnegative matrices, the assumption $A^k > 0$ means that the dominant eigenvalue of A^k is positive and simple while the corresponding eigenvector is positive. It is well known that the matrix A has as eigenvalues the k^{th} roots of those of A^k with the same eigenvectors. Since it happens $\forall k \geq k_0$, A possesses the strong Perron-Frobenius property. □

Theorem 2.2 *For a matrix $A \in I\!R^{n,n}$ the following properties are equivalent:*
i) Both matrices A and A^T possess the strong Perron-Frobenius property.
ii) There exists an integer $k_0 > 0$ such that $A^k > 0$ for all $k \geq k_0$.

Proof: ($i \Rightarrow ii$): Let $A = XDX^{-1}$ be the Jordan canonical form of the matrix A. We assume that the simple eigenvalue $\lambda_1 = \rho(A)$ is the first diagonal entry of D. So the Jordan canonical form can be written as

$$A = [x^{(1)} | X_{n,n-1}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D_{n-1,n-1} \end{array} \right] \left[\begin{array}{c} y^{(1)^T} \\ \hline Y_{n-1,n} \end{array} \right], \tag{2.1}$$

where $y^{(1)^T}$ and $Y_{n-1,n}$ are the first row and the matrix formed by the last $n-1$ rows of X^{-1}, respectively. Since A possesses the strong Perron-Frobenius property, the eigenvector $x^{(1)}$ is positive. From (2.1), the block form of A^T is

$$A^T = [y^{(1)} | Y^T_{n-1,n}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D^T_{n-1,n-1} \end{array} \right] \left[\begin{array}{c} x^{(1)^T} \\ \hline X^T_{n,n-1} \end{array} \right]. \tag{2.2}$$

The matrix $D^T_{n-1,n-1}$ is the block diagonal matrix formed by the transposes of all Jordan blocks except λ_1. It is obvious that there exists a permutation matrix $P \in I\!R^{n-1,n-1}$ such that the associated permutation transformation on the matrix $D^T_{n-1,n-1}$ transposes all the Jordan blocks. So, $D_{n-1,n-1} = P^T D^T_{n-1,n-1} P$ and relation (2.2) takes the form:

$$\begin{aligned} A^T &= [y^{(1)} | Y^T_{n-1,n}] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array} \right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^T \end{array} \right] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D^T_{n-1,n-1} \end{array} \right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array} \right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^T \end{array} \right] \left[\begin{array}{c} x^{(1)^T} \\ \hline X^T_{n,n-1} \end{array} \right] \\ &= [y^{(1)} | Y'^T_{n-1,n}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D_{n-1,n-1} \end{array} \right] \left[\begin{array}{c} x^{(1)^T} \\ \hline X'^T_{n,n-1} \end{array} \right], \end{aligned} \tag{2.3}$$

where $Y'^T_{n-1,n} = Y^T_{n-1,n}P$ and $X'^T_{n,n-1} = P^T X^T_{n,n-1}$. The last relation is the Jordan canonical form of A^T which means that $y^{(1)}$ is the eigenvector corresponding to the dominant eigenvalue λ_1. Since A^T possesses the strong Perron-Frobenius property, $y^{(1)}$ is a positive vector or a negative one. Since $y^{(1)^T}$ is the first row of X^{-1} we have that $(y^{(1)}, x^{(1)}) = 1$ implying that $y^{(1)}$ is a positive vector.
We return now to the Jordan canonical form (2.1) of A and form the power A^k

$$A^k = [x^{(1)}|X_{n,n-1}] \left[\begin{array}{c|c} \lambda_1^k & 0 \\ \hline 0 & D^k_{n-1,n-1} \end{array}\right] \left[\begin{array}{c} y^{(1)^T} \\ \hline Y_{n-1,n} \end{array}\right]$$

or

$$\frac{1}{\lambda_1^k} A^k = [x^{(1)}|X_{n,n-1}] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \frac{1}{\lambda_1^k} D^k_{n-1,n-1} \end{array}\right] \left[\begin{array}{c} y^{(1)^T} \\ \hline Y_{n-1,n} \end{array}\right].$$

Since λ_1 is the simple dominant eigenvalue, we get that

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} D^k_{n-1,n-1} = 0.$$

So,

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} A^k = x^{(1)} y^{(1)^T} > 0.$$

The last relation means that there exists an integer $k_0 > 0$ such that $A^k > 0$ for all $k \geq k_0$ and the first part of Theorem is proved.

($ii \Rightarrow i$): From the Perron-Frobenius theory of nonnegative matrices, the assumption $A^k > 0$ means that the dominant eigenvalue of A^k is positive and simple while the corresponding eigenvector is positive. Considering the Jordan canonical form of A^k, $\forall k \geq k_0$, we get that the matrix A has as the dominant eigenvalue the positive k^{th} root of the one of A^k with the same eigenvector. So, A possesses the strong Perron-Frobenius property. The proof for the matrix A^T is the same by taking $(A^k)^T = (A^T)^k > 0$. □

We observe that Theorem 2.1 is a special case of Theorem 2.2. Nevertheless, it is stated and proved since the proof is quite different and easier than that of Theorem 2.2.

In the sequel some statements with necessary conditions only follow.

Theorem 2.3 *If $A^T \in \mathbb{R}^{n,n}$ possesses the Perron-Frobenius property, then either*

$$\sum_{j=1}^{n} a_{ij} = \rho(A) \quad \forall\, i = 1(1)n, \tag{2.4}$$

or

$$\min_i \left(\sum_{j=1}^{n} a_{ij} \right) \leq \rho(A) \leq \max_i \left(\sum_{j=1}^{n} a_{ij} \right). \tag{2.5}$$

Moreover, if A^T possesses the strong Perron-Frobenius property, then both inequalities in (2.5) are strict.

Proof: Let that $(\rho(A), y)$ is the Perron-Frobenius eigenpair of the matrix A^T and $\xi \in \mathbb{R}^n$ is the vector of ones $\left(\xi = (1\ 1\ \cdots\ 1)^T\right)$. We form the product $y^T A\xi$:

$$y^T A\xi = y^T \begin{pmatrix} \sum_{j=1}^n a_{1j} \\ \sum_{j=1}^n a_{2j} \\ \vdots \\ \sum_{j=1}^n a_{nj} \end{pmatrix} = \sum_{i=1}^n \left(y_i \sum_{j=1}^n a_{ij} \right) \leq \max_i \left(\sum_{j=1}^n a_{ij} \right) \sum_{i=1}^n y_i. \tag{2.6}$$

Similarly, we have that

$$y^T A\xi = \sum_{i=1}^n \left(y_i \sum_{j=1}^n a_{ij} \right) \geq \min_i \left(\sum_{j=1}^n a_{ij} \right) \sum_{i=1}^n y_i. \tag{2.7}$$

On the other hand we get

$$y^T A\xi = \xi^T A^T y = \rho(A)\xi^T y = \rho(A) \sum_{i=1}^n y_i. \tag{2.8}$$

Relations (2.6), (2.7) and (2.8) give us relation (2.5). It is obvious that the inequalities in (2.5) become equalities if $\max_i \left(\sum_{j=1}^n a_{ij}\right) = \min_i \left(\sum_{j=1}^n a_{ij}\right)$, which proves the equality (2.4). It is also obvious that the inequalities in (2.6) and (2.7) become strict if $y > 0$. So, the inequalities in (2.5) become strict if A^T possesses the strong Perron-Frobenius property. □

Note that it is necessary to have $\max_i \left(\sum_{j=1}^n a_{ij}\right) > 0$, otherwise Theorem 2.3 does not hold and so, A^T does not possess the Perron-Frobenius property. On the other hand, it is not necessary to have $\min_i \left(\sum_{j=1}^n a_{ij}\right) \geq 0$ as is shown in the following example.

Example 2.1 Let

$$A = \begin{pmatrix} 1 & 1 & -3 \\ -4 & 1 & 1 \\ 8 & 5 & 8 \end{pmatrix}.$$

The vector of the row sums of A is $(-1\ \ -2\ \ 21)^T$, while A^T possesses the strong Perron-Frobenius property with the Perron-Frobenius eigenpair: $\left(6.868\ ,\ (0.4492\ \ 0.6225\ \ 0.6408)^T\right)$.

By interchanging the roles of A and A^T, Theorem 2.3 gives an analogous result for the column sums. This is presented in the following corollary.

Corollary 2.1 *If $A \in \mathbb{R}^{n,n}$ possesses the Perron-Frobenius property, then either*

$$\sum_{i=1}^n a_{ij} = \rho(A) \quad \forall\, j = 1(1)n, \tag{2.9}$$

or

$$\min_j \left(\sum_{i=1}^n a_{ij} \right) \leq \rho(A) \leq \max_j \left(\sum_{i=1}^n a_{ij} \right). \tag{2.10}$$

Moreover, if A possesses the strong Perron-Frobenius property, then both inequalities in (2.10) are strict.

We define now the space $\mathcal{P}$ of all vectors $x \geq 0$ with at least one component being positive and its subspace $\mathcal{P}^*$, the hyperoctant of vectors $x > 0$. Then, the previous results are generalized as follows.

Theorem 2.4 *If $A^T \in I\!R^{n,n}$ possesses the Perron-Frobenius property and $x \in \mathcal{P}^*$, then either*

$$\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} = \rho(A) \quad \forall\ i = 1(1)n, \tag{2.11}$$

or

$$\min_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \right) \leq \rho(A) \leq \max_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \right). \tag{2.12}$$

Moreover, if A^T possesses the strong Perron-Frobenius property, then both inequalities in (2.12) are strict and

$$\sup_{x\in\mathcal{P}^*} \left\{ \min_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \right) \right\} = \rho(A) = \inf_{x\in\mathcal{P}^*} \left\{ \max_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \right) \right\}. \tag{2.13}$$

Proof: Let $x \in \mathcal{P}^*$. We define the diagonal matrix $D = \operatorname{diag}(x_1, x_2, \cdots, x_n)$ and consider the similarity transformation $B = D^{-1}AD$ (see Varga [16], Theorem 2.2). Then the entries of B are $b_{ij} = \frac{a_{ij}x_j}{x_i}$. Since B is produced from A by a similarity transformation and D and D^{-1} are both nonnegative matrices, we obtain that B^T possesses also the Perron-Frobenius property. As a consequence we have

$$\sup_{x\in\mathcal{P}^*} \left\{ \min_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \right) \right\} \leq \rho(A) \leq \inf_{x\in\mathcal{P}^*} \left\{ \max_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \right) \right\}, \tag{2.14}$$

which implies (2.12). We choose now the Perron-Frobenius eigenvector y in the place of x. It is easily seen that inequalities (2.12) become equalities, which means that those in (2.14) become also equalities and the proof is complete. □

By interchanging the roles of A and A^T, Theorem 2.4 gives us analogous results for the column sums stated in the corollary below.

Corollary 2.2 *If $A \in I\!R^{n,n}$ possesses the Perron-Frobenius property and $x \in \mathcal{P}^*$, then either*

$$\frac{\sum_{j=1}^n a_{ji}x_j}{x_i} = \rho(A) \quad \forall\ i = 1(1)n, \tag{2.15}$$

or

$$\min_i \left(\frac{\sum_{j=1}^n a_{ji}x_j}{x_i} \right) \leq \rho(A) \leq \max_i \left(\frac{\sum_{j=1}^n a_{ji}x_j}{x_i} \right). \tag{2.16}$$

Moreover, if A^T possesses the strong Perron-Frobenius property, then both inequalities in (2.12) are strict and

$$\sup_{x\in\mathcal{P}^*} \left\{ \min_i \left(\frac{\sum_{j=1}^n a_{ji}x_j}{x_i} \right) \right\} = \rho(A) = \inf_{x\in\mathcal{P}^*} \left\{ \max_i \left(\frac{\sum_{j=1}^n a_{ji}x_j}{x_i} \right) \right\}. \tag{2.17}$$

In the sequel we give some monotonicity properties concerning the dominant eigenvalue in the case where the matrices possess the Perron-Frobenius property. It is well known that the eigenvalues and the entries of the eigenvectors are continuous functions of the entries of a matrix A. So, if A possesses the strong Perron-Frobenius property, then a perturbation of A, $\widetilde{A} = A + E$ provided $\|E\|$ is small enough, possesses also the strong Perron-Frobenius property. It is also well known, from the theory of nonnegative matrices, that the dominant eigenvalue of a nonnegative matrix A is a nondecreasing function of the entries of A, when A is reducible, while if A is an irreducible matrix, it is a strictly increasing function. Then two questions come up: What happens to the monotonicity in case the matrices possess the Perron-Frobenius property? Does the property of "possessing the Perron-Frobenius property" still hold when the entries of A increase, as it does in the nonnegative case? Unfortunately, the answer to the second question is not positive. It depends on the direction in which we increase the entries, as we will see later. First we give some properties which provide an answer to the first question.

Theorem 2.5 *If the matrices $A, B \in I\!R^{n,n}$ are such that $A \leq B$, and both A and B^T possess the Perron-Frobenius property (or both A^T and B possess the Perron-Frobenius property), then*

$$\rho(A) \leq \rho(B). \tag{2.18}$$

Moreover, if the above matrices possess the strong Perron-Frobenius property and $A \neq B$, then the inequality in (2.18) is strict.

Proof: Let $x \geq 0$ be the Perron-Frobenius eigenvector of A associated with the dominant eigenvalue λ_A and let $y \geq 0$ be the Perron-Frobenius eigenvector of B^T associated with the dominant eigenvalue λ_B. Then the following equalities hold

$$y^T Ax = \lambda_A y^T x, \quad y^T Bx = \lambda_B y^T x.$$

Since $A \leq B$, we can write $B = A + C$, where $C \geq 0$. So,

$$y^T Bx = y^T (A + C)x = y^T Ax + y^T Cx \geq y^T Ax.$$

Assuming that $y^T x > 0$, the above relations imply that $\lambda_B \geq \lambda_A$. The case where $y^T x = 0$ is covered by using a continuity argument. For this we consider the matrices A' and B' which are small perturbations of the matrices A and B, respectively, such that for the corresponding perturbed eigenvectors we will have $y'^T x' > 0$. The above inequality holds for the perturbed eigenvalues and because of the continuity the same property holds for the eigenvalues of A and B. It is obvious that if we follow the same reasoning we can obtain the same result in case both A^T and B possess the Perron-Frobenius property. It is also obvious that the inequality becomes strict in case the associated Perron-Frobenius properties are strong. □

We note that the above property does not guarantee the existence of the Perron-Frobenius property for an intermediate matrix C ($A \leq C \leq B$) and does not give any information about $\rho(C)$.

Theorem 2.6 *Let (i) $A^T \in \mathbb{R}^{n,n}$ possess the Perron-Frobenius property and $x \geq 0$ $(x \neq 0)$ be such that $Ax - \alpha x \geq 0$ for a constant $\alpha > 0$ or (ii) $A \in \mathbb{R}^{n,n}$ possess the Perron-Frobenius property and $x \geq 0$ $(x \neq 0)$ be such that $x^T A - \alpha x^T \geq 0$ for a constant $\alpha > 0$. Then*

$$\alpha \leq \rho(A). \tag{2.19}$$

Moreover, if $Ax - \alpha x > 0$ or $x^T A - \alpha x^T > 0$, then the inequality in (2.19) is strict.

Proof: For hypothesis (i), let $y \geq 0$ be the Perron-Frobenius eigenvector of A associated with $\rho(A)$. Then, the following equivalence holds

$$y^T(Ax - \alpha x) \geq 0 \Longleftrightarrow (\rho(A) - \alpha)y^T x \geq 0.$$

If $y^T x > 0$, then the inequality (2.19) holds. In the case where $y^T x = 0$ we recall the perturbation argument used in Theorem 2.5 to prove the validity of (2.19). If $Ax - \alpha x > 0$, the above inequalities become strict and therefore (2.19) becomes strict. For hypothesis (ii) the proof is similar. □

The above theorem is an extension of Corollary 3.2 given by Marek and Szyld in [8], for nonnegative matrices. The following theorem is also an extension of Lemma 3.3 of the same paper [8].

Theorem 2.7 *Let (i) $A^T \in \mathbb{R}^{n,n}$ possess the Perron-Frobenius property and $x > 0$ be such that $\alpha x - Ax \geq 0$ for a constant $\alpha > 0$ or (ii) $A \in \mathbb{R}^{n,n}$ possess the Perron-Frobenius property and $x > 0$ be such that $\alpha x^T - x^T A \geq 0$ for a constant $\alpha > 0$. Then*

$$\rho(A) \leq \alpha. \tag{2.20}$$

Moreover, if $\alpha x - Ax > 0$ or $\alpha x^T - x^T A > 0$, then the inequality in (2.20) becomes strict.

Proof: As in the previous theorem we give the proof only for hypothesis (i). Let $y \geq 0$ be the Perron-Frobenius eigenvector of A associated with $\rho(A)$. Then, we have

$$y^T(\alpha x - Ax) \geq 0 \Longleftrightarrow (\alpha - \rho(A))y^T x \geq 0.$$

Since $x > 0$ we have that $y^T x > 0$ and the inequality (2.20) holds. If $\alpha x - Ax > 0$, the above inequalities become strict and therefore (2.20) becomes strict. □

We remark that the condition $x > 0$ is necessaty. This is because for $x \geq 0$ such that $Ax = 0$, the condition $\alpha x - Ax \geq 0$ holds for any $\alpha \geq 0$, but the inequality (2.20) is not true for any $\alpha \geq 0$.

We give now two monotonicity properties depending on the direction in which the entries of a matrix increase.

Theorem 2.8 *Let $A \in I\!R^{n,n}$ possess the Perron-Frobenius property with $x \geq 0$ the associated eigenvector. Then, for the matrix B such that*

$$B = A + \epsilon x y^T, \quad \epsilon > 0, \; y \geq 0 \tag{2.21}$$

there holds

$$\rho(A) \leq \rho(B). \tag{2.22}$$

Moreover, if A possesses the strong Perron-Frobenius property and $y \geq 0$ ($y \neq 0$), then inequality in (2.22) becomes strict.

Proof: By post-multiplying (2.21) by x we obtain

$$Bx = (A + \epsilon x y^T)x = \left(\rho(A) + \epsilon y^T x\right) x$$

which means that $\rho(A) + \epsilon y^T x$ is an eigenvalue of B. Since $\epsilon y^T x \geq 0$ we take the inequality (2.22). The analogous proof for the strict case is obvious and is omitted. □

It is obvious that an analogous property could be given by considering that A^T possesses the Perron-Frobenius property. However, we have to remark that the above property does not guarantee the existence of the Perron-Frobenius property for the matrix B. To do this we give the following statement.

Theorem 2.9 *Let $A \in I\!R^{n,n}$ be such that both A and A^T possess the strong Perron-Frobenius property with x and y being the associated eigenvectors, respectively. Then, for the matrix B such that*

$$B = A + \epsilon x y^T, \quad \epsilon > 0, \tag{2.23}$$

there holds that both B and B^T possess the strong Perron-Frobenius property and

$$\rho(A) < \rho(B). \tag{2.24}$$

Proof: The proof of the strict inequality (2.24) is obtained from Theorem 2.8 and from the fact that $x, y > 0$. To prove the existence of the strong Perron-Frobenius property of B and B^T we use Theorem 2.2. We form $B^k = (A + \epsilon x y^T)^k$ and expand it into a sum of products of the matrices A and xy^T with the first term being A^k. Since $Axy^T = \rho(A)xy^T$ and $xy^T A = \rho(A)xy^T$, all the other $2^k - 1$ terms in the expansion, except A^k, are eventually positive scalar multiples of powers of the matrix xy^T. This means that the sum of all the other terms, except the first one, is a positive matrix. From Theorem 2.2 we have that there exists a k_0 such that $A^k > 0$ for all $k \geq k_0$. So, for this k_0 we have also $B^k > 0$ for all $k \geq k_0$, which means that both B and B^T possess the strong Perron-Frobenius property. □

We have to remark here that Theorem 2.8 gives a weak result for a dense set of directions xy^T, for all $y \geq 0$, while Theorem 2.9 gives a stronger result for precisely one direction xy^T. Based on continuity properties we can conclude that the last result is valid also for a cone of directions around xy^T.

3 Convergence theory of Perron-Frobenius splittings

In this section we define first the Perron-Frobenius splittings analogous to Regular, Weak Regular and Nonnegative splittings.

Definition 3.1 *Let $A \in I\!R^{n,n}$ be a nonsigular matrix. The splitting $A = M - N$ is*
(i) a Perron-Frobenius splitting of the first kind (kind I) if $M^{-1}N$ possesses the Perron-Frobenius property.
(ii) a Perron-Frobenius splitting of the second kind (kind II) if NM^{-1} possesses the Perron-Frobenius property.

In the sequel, for simplicity, by the term *Perron-Frobenius splitting* we mean Perron-Frobenius splitting of kind I. It is obvious from the above definition that the classes of Regular splittings, Weak Regular splittings and Nonnegative splittings belong to the class of Perron-Frobenius splittings. So, the class of Perron-Frobenius splittings is an extension of the well known, previously defined, classes. In the following, we state and prove convergence and comparison statements about this new class of splittings.

3.1 Convergence Theorems

The following theorem is an extension of the one given by Climent and Perea [4].

Theorem 3.1 *Let $A \in I\!R^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ be a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector. Then the following properties are equivalent:*
(i) $\rho(M^{-1}N) < 1$
(ii) $A^{-1}N$ possesses the Perron-Frobenius property
(iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$
(iv) $A^{-1}Mx \geq x$
(v) $A^{-1}Nx \geq M^{-1}Nx$.

Proof: It can be readily found out that the matrices $A^{-1}N$ and $M^{-1}N$ are connected via the relations yielded below.

$$A^{-1}N = (M - N)^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N \tag{3.25}$$

or

$$M^{-1}N = (A + N)^{-1}N = (I + A^{-1}N)^{-1}A^{-1}N. \tag{3.26}$$

The above relations imply that the matrices $A^{-1}N$ and $M^{-1}N$ have the same sets of eigenvectors with their eigenvalues being connected by

$$\mu_i = \frac{\lambda_i}{1 - \lambda_i}, \quad i = 1, 2, \cdots, n, \tag{3.27}$$

where $\lambda_i, \mu_i,\ i = 1, 2, \cdots, n$, are the eigenvalues of $M^{-1}N$ and $A^{-1}N$, respectively.

(i) $\Longrightarrow$ (ii): From $\rho(M^{-1}N) < 1$ and (3.27), there is an eigenvalue $\mu = \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)} > 0$ of $A^{-1}N$ corresponding to the eigenvector x. Looking for a contradiction, assume that there is another eigenvalue $\mu' = \frac{\lambda'}{1-\lambda'}$ corresponding to $\rho(A^{-1}N)$. So,

$$\rho(A^{-1}N) = |\mu'| = \frac{|\lambda'|}{|1-\lambda'|} > \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)} = |\mu|.$$

The eigenvalue λ' belongs to the disc $|z| \leq \rho(M^{-1}N)$ and $1-\rho(M^{-1}N)$ is the distance of the point 1 from this disc. So, $|1-\lambda'| \geq 1-\rho(M^{-1}N)$ which constitutes a contradiction.

(ii) $\Longrightarrow$ (iii): Since $A^{-1}N$ has the Perron-Frobenius eigenpair $(\rho(A^{-1}N), x)$, property (iii) follows from (3.26) by a post-multiplication by x.

(iii) $\Longrightarrow$ (i): It holds because $\rho(A^{-1}N) > 0$.

(i) $\Longleftrightarrow$ (iv): It is obvious that

$$A^{-1}Mx = (M-N)^{-1}Mx = (I-M^{-1}N)^{-1}x = \frac{1}{1-\rho(M^{-1}N)}x.$$

Since $x \geq 0,\ x \neq 0$,

$$\frac{1}{1-\rho(M^{-1}N)}x \geq x \Longleftrightarrow 0 < 1-\rho(M^{-1}N) < 1 \Longleftrightarrow 0 < \rho(M^{-1}N) < 1.$$

(i) $\Longleftrightarrow$ (v): Considering relation (3.25) and the fact that $x \geq 0,\ x \neq 0$, we get

$$A^{-1}Nx \geq M^{-1}Nx \Longleftrightarrow \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)}x \geq \rho(M^{-1}N)x \Longleftrightarrow \rho(M^{-1}N) < 1.$$

□

We can also state an analogous Theorem for the convergence properties of the Perron-Frobenius splittings of kind II. The proof follows the same lines as before and is omitted.

Theorem 3.2 *Let $A \in \mathbb{R}^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ be a Perron-Frobenius splitting of kind II, with x the Perron-Frobenius eigenvector. Then the following properties are equivalent:*
(i) $\rho(M^{-1}N) = \rho(NM^{-1}) < 1$
(ii) NA^{-1} *possesses the Perron-Frobenius property*
(iii) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$
(iv) $MA^{-1}x \geq x$
(v) $NA^{-1}x \geq NM^{-1}x$.

Theorems 3.1 and 3.2 give sufficient and necessary conditions for a Perron-Frobenius splitting to be convergent. The following two theorems give only sufficient convergence conditions and constitute also extensions of the ones given by Climent and Perea [4].

Theorem 3.3 *Let $A \in \mathbb{R}^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ is a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector. If one of the following properties holds true:*
(i) There exists $y \in \mathbb{R}^n$ such that $A^Ty \geq 0$, $N^Ty \geq 0$ and $y^TAx > 0$
(ii) There exists $y \in \mathbb{R}^n$ such that $A^Ty \geq 0$, $M^Ty \geq 0$ and $y^TAx > 0$
then $\rho(M^{-1}N) < 1$.

Proof: We consider the vector z such that $y = (A^T)^{-1}z$, then the above properties are modified as follows:
(i) There exists $z \geq 0$ such that $z^T(A^{-1}N) \geq 0$, $z^Tx > 0$, and
(ii) There exists $z \geq 0$ such that $z^T(A^{-1}M) \geq 0$, $z^Tx > 0$,
respectively. We suppose that property (i) holds true. By post-multiplying by x we get

$$z^T(A^{-1}N)x = \mu z^T x \geq 0,$$

where μ is the eigenvalue of $A^{-1}N$ corresponding to the eigenvector x. So, $\mu = \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)}$. Since $z^Tx > 0$ we get that $\mu \geq 0$, which means that $\rho(M^{-1}N) < 1$.

Let that property (ii) holds true, then by following the same steps we get

$$z^T(A^{-1}M)x = \mu' z^T x > 0$$

where $\mu' = \frac{1}{1-\rho(M^{-1}N)} > 0$ which leads to the same result. □

Moreover, we can prove that property (ii) is stronger than property (i), which means that the validity of (i) implies the validity of (ii) but the converse is not true. For this let that property (i) holds. Then

$$A^T y \geq 0 \Longrightarrow M^T y - N^T y \geq 0 \Longrightarrow M^T y \geq N^T y \geq 0.$$

it is obvious that the converse cannot hold.

For the Perron-Frobenius splittings of kind II, the following theorem is stated.

Theorem 3.4 *Let $A \in \mathbb{R}^{n,n}$ be a nonsigular matrix and the splitting $A^T = M^T - N^T$ is a Perron-Frobenius splitting of kind II, with x the Perron-Frobenius eigenvector. If one of the following properties holds true:*
(i) There exists $y \in \mathbb{R}^n$ such that $Ay \geq 0$, $Ny \geq 0$ and $y^TA^Tx > 0$
(ii) There exists $y \in \mathbb{R}^n$ such that $Ay \geq 0$, $My \geq 0$ and $y^TA^Tx > 0$
then $\rho(M^{-1}N) < 1$.

We have to remark here that because of the sufficient conditions only, in Theorems 3.3 and 3.4, we cannot have any information about the convergence unless such a y vector exists. We show this by the following three examples.

Example 3.1

$$(i)\ A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix},\ N = \begin{pmatrix} -2 & 3 \\ -7 & 7 \end{pmatrix},\ M = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix},\ T = \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix},$$

$$A^{-1}N = \begin{pmatrix} -3 & 1 \\ -0.5 & -1 \end{pmatrix},\ A^{-1}M = \begin{pmatrix} -2 & 1 \\ -0.5 & 0 \end{pmatrix},\ \rho(T) = 4.4142,\ x = \begin{pmatrix} 0.5054 \\ 0.8629 \end{pmatrix},$$

where $T = M^{-1}N$. A vector $z \geq 0$ ($z \neq 0$) such that either $z^T(A^{-1}N) \geq 0$ or $z^T(A^{-1}M) \geq 0$ does not exist and so the splitting is **not** convergent.

$$(ii)\ A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix},\ N = \begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix},\ M = \begin{pmatrix} 0 & -2 \\ 8 & -5 \end{pmatrix},\ T = \begin{pmatrix} 0.9375 & -0.125 \\ 0.5 & 0 \end{pmatrix},$$

$$A^{-1}N = \begin{pmatrix} 7 & -1 \\ 4 & -0.5 \end{pmatrix}, \; A^{-1}M = \begin{pmatrix} 8 & -1 \\ 4 & 0.5 \end{pmatrix}, \; \rho(T) = 0.8653, \; x = \begin{pmatrix} 0.8658 \\ 0.5003 \end{pmatrix}.$$

There exists no $z \geq 0$ ($z \neq 0$) such that $z^T(A^{-1}N) \geq 0$ but for $z^T = (1 \;\; 3)$ we have $z^T(A^{-1}M) \geq 0$, so the splitting **is** convergent.

$$(iii)\; A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \; N = \begin{pmatrix} -1 & 0 \\ 5 & -3 \end{pmatrix}, \; M = \begin{pmatrix} 0 & -2 \\ 8 & -7 \end{pmatrix}, \; T = \begin{pmatrix} 1.0625 & -0.3750 \\ 0.5 & 0 \end{pmatrix},$$

$$A^{-1}N = \begin{pmatrix} 7 & -3 \\ 4 & -1.5 \end{pmatrix}, \; A^{-1}M = \begin{pmatrix} 8 & -3 \\ 4 & -0.5 \end{pmatrix}, \; \rho(T) = 0.8390, \; x = \begin{pmatrix} 0.8590 \\ 0.5119 \end{pmatrix}.$$

There exists no $z \geq 0$ ($z \neq 0$) such that either $z^T(A^{-1}N) \geq 0$ or $z^T(A^{-1}M) \geq 0$ but the splitting **is** convergent.

We have also to remark that the strict condition $y^T Ax > 0$ is necessary. This is shown in the following example.

Example 3.2

$$A = \begin{pmatrix} 1 & -2 & -1 \\ 3 & -4 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \; N = \begin{pmatrix} -2 & 3 & 1 \\ -7 & 7 & 1 \\ 2.5 & -2 & 1 \end{pmatrix}, \; M = \begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 2 \\ 1.5 & -1 & 2 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 2 & \frac{8}{3} \\ -1 & 5 & \frac{11}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix},$$

$$A^{-1}N = \begin{pmatrix} -3 & 1 & -2.5 \\ -0.5 & -1 & -2 \\ 0 & 0 & 0.5 \end{pmatrix}, \; \rho(T) = 4.4142, \; x = \begin{pmatrix} 0.5054 \\ 0.8629 \\ 0 \end{pmatrix}.$$

For the vector $z^T = (0 \;\; 0 \;\; 1)$ all but one of the conditions of Theorem 3.3 (i) hold. However, since $z^T x = 0$ the splitting is **not** convergent.

From Theorems 3.3 and 3.4 the corollaries below follow.

Corollary 3.1 *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix and the splitting $A = M - N$ be a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector. If one of the matrices $(A^{-1}N)^T$ or $(A^{-1}M)^T$ possesses also the Perron-Frobenius property with y the associated Perron-Frobenius eigenvector, such that $y^T x > 0$, then $\rho(M^{-1}N) < 1$.*

Proof: Since $y \geq 0$ and $y^T(A^{-1}N) \geq 0$ or $y^T(A^{-1}M) \geq 0$, respectively, the vector y plays the role of z in the proof of Theorem 3.3, so the splitting is convergent. □

Corollary 3.2 *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix and the splitting $A^T = M^T - N^T$ be a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector. If one of the matrices NA^{-1} or MA^{-1} possesses also the Perron-Frobenius property with y the associated Perron-Frobenius eigenvector, such that $y^T x > 0$, then $\rho(M^{-1}N) < 1$.*

3.2 Comparison Theorems

The following theorem is an extension of the one given by Marek and Szyld [8] for nonnegative splittings.

Theorem 3.5 *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix such that $A^{-1} \geq 0$. If one of the following properties holds true:*
(i) $A = M_1 - N_1$ and $A^T = M_2^T - N_2^T$ are two convergent Perron-Frobenius splittings of kind I and of kind II, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^T := (M_2^{-1}N_2)^T$ and $x \geq 0$, $y \geq 0$ the associated Perron-Frobenius eigenvectors, respectively, and

$$N_2 x \geq N_1 x, \tag{3.28}$$

(ii) $A^T = M_1^T - N_1^T$ and $A = M_2 - N_2$ are two convergent Perron-Frobenius splittings of kind II and of kind I, respectively, with $T_1^T := (M_1^{-1}N_1)^T$, $T_2 := M_2^{-1}N_2$ and $y' \geq 0$, $z > 0$ the associated Perron-Frobenius eigenvectors, respectively, and

$$N_2 z \geq N_1 z, \tag{3.29}$$

then

$$\rho(T_1) \leq \rho(T_2). \tag{3.30}$$

Moreover, if $A^{-1} > 0$ and $N_2 x \neq N_1 x$, $N_2 z \neq N_1 z$, respectively, then

$$\rho(T_1) < \rho(T_2). \tag{3.31}$$

Proof: Let that property (i) holds. Then

$$A^{-1}N_2 x \geq A^{-1}N_1 x.$$

Since the above splittings are convergent, from Theorem 3.1 property (ii), we get that the matrix $A^{-1}N_1$ possesses the Perron-Frobenius property and from Theorem 3.2 property (ii), we get that the matrix $(A^{-1}N_2)^T$ possesses the Perron-Frobenius property, with x and y the Perron-Frobenius eigenvectors, respectively. So,

$$A^{-1}N_2 x - \rho(A^{-1}N_1)x \geq 0$$

and by Theorem 2.6 we get that $\rho(A^{-1}N_2) \geq \rho(A^{-1}N_1)$. Since $\rho(A^{-1}N_1) = \frac{\rho(T_1)}{1-\rho(T_1)}$, $\rho((A^{-1}N_2)^T) = \rho(A^{-1}N_2) = \frac{\rho(T_2)}{1-\rho(T_2)}$ and the fact that the function $\frac{\rho}{1-\rho}$ is an increasing function of $\rho \in (0,1)$, the result (3.30) follows. The strict inequality (3.31) becomes obvious from the fact that $A^{-1} > 0$ and $N_2 x \neq N_1 x$, $N_2 z \neq N_1 z$, respectively. The proof in case property (ii) holds is analogous, where use of Theorem 2.7 is made this time. □

We show the validity of this theorem by the following example.

Example 3.3 We consider the splittings $A = M_1 - N_1 = M_2 - N_2 = M_3 - N_3$ where

$$A = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -2 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 4 & -1.1 & 0.2 & 0 \\ -1.1 & 4 & -1 & 0 \\ 0.2 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}.$$

The splitting $A = M_1 - N_1$ is a Perron-Frobenius splitting with the Perron-Frobenius eigenpair being $(\rho(T_1), x_1) = \left(0.5345, (0.5680\ 0.4212\ 0.4212\ 0.5680)^T\right)$. The splitting $A^T = M_2^T - N_2^T$ is a Perron-Frobenius splitting of kind II with the Perron-Frobenius eigenpair being $(\rho(T_2), y_2) = \left(0.6126, (0.6388\ 0.2855\ 0.3871\ 0.6005)^T\right)$. Although $N_2 - N_1$ is not a nonnegative matrix, we have $(N_2 - N_1)x_1 = (0.0421\ 0.3644\ 0.5348\ 0)^T \geq 0$. Moreover, $A^{-1} > 0$ and $N_2x_1 \neq N_1x_1$. So, property (i) of Theorem 3.5 holds and the inequality $\rho(T_1) < \rho(T_2)$ is confirmed. We can check that for the first two splittings, property (ii) of Theorem 3.5 also holds.
To compare the last two splittings we observe that the splitting $A = M_2 - N_2$ is a Perron-Frobenius splitting while $A = M_3 - N_3$ is a regular splitting, but properties (i) and (ii) of Theorem 3.5 do not hold. So, Theorem 3.5 does not give any information.
We have to observe here that both properties (i) and (ii) of Theorem 3.5 hold for the comparison of the first splitting with the last one, since $N_3 - N_1 \geq 0$. So, $\rho(T_1) = 0.5345 < \rho(T_3) = 0.6667$ is confirmed.

The above theorem can be extended further by replacing condition $A^{-1} \geq 0$ by a weaker one. So, we can have the following statement.

Theorem 3.6 *Let $A \in I\!R^{n,n}$ be a nonsingular matrix. If one of the following properties holds true:*
(i) $A = M_1 - N_1$ and $A^T = M_2^T - N_2^T$ are two convergent Perron-Frobenius splittings of kind I and of kind II, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^T := (M_2^{-1}N_2)^T$ and $x \geq 0$, $y \geq 0$ the associated Perron-Frobenius eigenvectors, respectively, such that

$$y^T A^{-1} \geq 0, \quad y^T x > 0 \quad \textit{and} \quad N_2 x \geq N_1 x, \tag{3.32}$$

(ii) $A^T = M_1^T - N_1^T$ and $A = M_2 - N_2$ are two convergent Perron-Frobenius splittings of kind II and of kind I, respectively, with $T_1^T := (M_1^{-1}N_1)^T$, $T_2 := M_2^{-1}N_2$ and $y' \geq 0$, $z > 0$ the associated Perron-Frobenius eigenvectors, respectively, such that

$$y'^T A^{-1} \geq 0, \quad y'^T z > 0 \quad \textit{and} \quad N_2 z \geq N_1 z, \tag{3.33}$$

then

$$\rho(T_1) \leq \rho(T_2). \tag{3.34}$$

Moreover, if $y^T A^{-1} > 0$ and $N_2x \neq N_1x$ for property (i) or $y'^T A^{-1} > 0$ and $N_2z \neq N_1z$ for property (ii) , then

$$\rho(T_1) < \rho(T_2). \tag{3.35}$$

Proof: Let that property (i) holds. Then from the first and the last inequalities of (3.32) we get

$$y^T A^{-1} N_2 x \geq y^T A^{-1} N_1 x.$$

As in Theorem 3.5, it can be implied in a similar way that both matrices $A^{-1}N_1$ and $(A^{-1}N_2)^T$ possess the Perron-Frobenius property, with x and y the Perron-Frobenius eigenvectors, respectively. So,

$$\rho(A^{-1}N_2)y^Tx - \rho(A^{-1}N_1)y^Tx \geq 0$$

and therefore $\rho(T_1) \leq \rho(T_2)$. The strict inequality (3.35) is obvious. The proof in case property (ii) holds is similar. □

Theorem 3.7 *Let $A \in I\!R^{n,n}$ be a nonsingular matrix. If one of the following properties holds true:*
(i) $A = M_1 - N_1$ and $A^T = M_2^T - N_2^T$ be two Perron-Frobenius convergent splittings of kind I and of kind II, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^T := (M_2^{-1}N_2)^T$ and $x \geq 0$, $y \geq 0$ the associated Perron-Frobenius eigenvectors, respectively,

$$N_1x \geq 0 \text{ and } M_1^{-1} \geq M_2^{-1}, \tag{3.36}$$

(ii) $A^T = M_1^T - N_1^T$ and $A = M_2 - N_2$ be two Perron-Frobenius convergent splittings of kind II and of kind I, respectively, with $T_1^T := (M_1^{-1}N_1)^T$, $T_2 := M_2^{-1}N_2$ and $y' \geq 0$, $z > 0$ the associated Perron-Frobenius eigenvectors, respectively,

$$N_2z \geq 0 \text{ and } M_1^{-1} \geq M_2^{-1}, \tag{3.37}$$

then

$$\rho(T_1) \leq \rho(T_2). \tag{3.38}$$

Moreover, if $M_1^{-1} > M_2^{-1}$ and $N_1x \neq 0$, $N_2z \neq 0$, respectively, then

$$\rho(T_1) < \rho(T_2). \tag{3.39}$$

Proof: We assume that property (i) holds. Then

$$M_1x = \frac{1}{\rho(T_1)}N_1x \geq 0$$

and

$$Ax = M_1(I - T_1)x = \frac{1-\rho(T_1)}{\rho(T_1)}N_1x \geq 0.$$

By premultiplying by $M_1^{-1} - M_2^{-1} \geq 0$ we get

$$(M_1^{-1} - M_2^{-1})Ax = (I - T_1)x - (I - T_2)x = T_2x - \rho(T_1)x \geq 0.$$

By Theorem 2.6 we obtain the result (3.38). The strict inequality (3.39) is obvious and that the proof in case property (ii) holds is quite analogous. □

We observe that Theorem 3.7 provides an answer to Example 3.3 where Theorem 3.5 failed. Especially, we have $M_2^{-1} - M_3^{-1} > 0$ and $N_2x_2 \geq 0$, $N_2x_2 \neq 0$. So the strict inequality $\rho(T_2) = 0.6126 < \rho(T_3) = 0.6667$ is confirmed. It is easily checked that property (ii) of

Theorem 3.7 also holds. We also observe that both properties (i) and (ii) of Theorem 3.7 hold for the comparison of the first with the second splitting as well as the first with the last one.

As we provided an extension from Theorem 3.5 to Theorem 3.6 we can extend also Theorem 3.7 by simply replacing the condition $M_1^{-1} \geq M_2^{-1}$ by a weaker one. This is stated in the following theorem, where the proof is similar to the previous one.

Theorem 3.8 *Let $A \in I\!R^{n,n}$ be a nonsingular matrix. If one of the following holds:*
(i) $A = M_1 - N_1$ and $A^T = M_2^T - N_2^T$ are two convergent Perron-Frobenius splittings of kind I and of kind II, respectively, with $T_1 := M_1^{-1}N_1$, $T_2^T := (M_2^{-1}N_2)^T$ and $x \geq 0$, $y \geq 0$ the associated Perron-Frobenius eigenvectors, respectively, $N_1x \geq 0$ and $y^TM_1^{-1} \geq y^TM_2^{-1}$, $y^Tx > 0$,
(ii) $A^T = M_1^T - N_1^T$ and $A = M_2 - N_2$ are two convergent Perron-Frobenius splittings of kind II and of kind I, respectively, with $T_1^T := (M_1^{-1}N_1)^T$, $T_2 := M_2^{-1}N_2$ and $y' \geq 0$, $z > 0$ the associated Perron-Frobenius eigenvectors, respectively, $N_2z \geq 0$ and $y'^TM_1^{-1} \geq y'^TM_2^{-1}$, $y'^Tx > 0$, then

$$\rho(T_1) \leq \rho(T_2). \tag{3.40}$$

Moreover, if $y^TM_1^{-1} > y^TM_2^{-1}$ and $N_1x \neq 0$ or $y'^TM_1^{-1} > y'^TM_2^{-1}$ and $N_2z \neq 0$, respectively, then the inequality (3.40) is strict, while if $y^TM_1^{-1} = y^TM_2^{-1}$ or $y'^TM_1^{-1} = y'^TM_2^{-1}$, respectively, then the inequality (3.40) becomes an equality.

In the following example it is shown how the three previous theorems work.

Example 3.4 We consider the splittings $A = M_1 - N_1 = M_2 - N_2 = M_3 - N_3 = M_4 - N_4 = M_5 - N_5$ where

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 3 & -1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

It is easily checked that all the above splittings are convergent ones with

$$\rho(T_2) = 0 < \rho(T_1) = \rho(T_3) = \rho(T_4) = \frac{1}{3} < \rho(T_5) = 0.4472.$$

The first four splittings are Perron-Frobenius splittings while the last one is a nonnegative splitting. The splittings $A^T = M_1^T - N_1^T = M_3^T - N_3^T = M_4^T - N_4^T$ are also Perron-Frobenius splittings while the splitting $A^T = M_5^T - N_5^T$ is a nonnegative splitting. The associated Perron-Frobenius eigenvectors are:

$$x_1 = x_2 = \begin{pmatrix} 0.7071 \\ 0.7071 \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0.8018 \\ 0.5345 \\ 0.2773 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{pmatrix}, \quad x_5 = \begin{pmatrix} 0.6325 \\ 0.7071 \\ 0.3162 \end{pmatrix},$$

$$y_1 = y_3 = y_4 = \begin{pmatrix} 0 \\ 0.7071 \\ 0.7071 \end{pmatrix}, \quad y_5 = \begin{pmatrix} 05130 \\ 0.6882 \\ 0.5130 \end{pmatrix},$$

where by x_i and y_i we have denoted the associated Perron-Frobenius eigenvectors of kind I and of kind II, respectively. It is easily checked that A^{-1} is not a nonnegative matrix so, Theorem 3.5 cannot be applied and therefore we will try to confirm our results by applying Theorems 3.6, 3.7 or 3.8. We use the symbol $i \leftrightarrow j$ to denote the comparison of the i^{th} splitting with the j^{th} one:

$1 \leftrightarrow 2$: It is easily checked that assumptions (i) of Theorems 3.6, 3.7 and 3.8 hold, where the roles of T_1 and T_2 have been interchanged, to obtain $\rho(T_2) \leq \rho(T_1)$. Note that the strict inequality cannot be obtained from any of the above theorems.

$1 \leftrightarrow 3$: Theorems 3.6 and 3.7 cannot by applied while both assumptions (i) and (ii) of Theorem 3.8 hold with the corresponding inequalities $y_3^T M_1^{-1} \geq y_3^T M_3^{-1}$ and $y_1^T M_1^{-1} \geq y_1^T M_3^{-1}$ being equalities. So, we obtain $\rho(T_1) = \rho(T_3)$.

$3 \leftrightarrow 2$: The same properties, as in the case $1 \leftrightarrow 2$, hold. Therefore, $\rho(T_2) \leq \rho(T_3)$.

$3 \leftrightarrow 4$: The same properties, as in the case $1 \leftrightarrow 3$, hold. So, $\rho(T_3) = \rho(T_4)$.

$4 \leftrightarrow 2$: The same properties, as in the case $1 \leftrightarrow 2$, hold. Consequently, $\rho(T_2) \leq \rho(T_4)$.

$4 \leftrightarrow 5$: Both properties of Theorems 3.6, 3.7 and 3.8 are applied to give the inequality $\rho(T_4) \leq \rho(T_5)$. Moreover, we have that $y_5^T A^{-1} > 0$ and $y_5^T M_4^{-1} > y_5^T M_5^{-1}$, which gives by Theorems 3.6 and 3.8, respectively, the strict inequality $\rho(T_4) < \rho(T_5)$.

$5 \leftrightarrow 2$: From property (i) of Theorem 3.6 and the fact that $y_5^T A^{-1} > 0$ we obtain the strict inequality $\rho(T_2) < \rho(T_5)$.

We conclude this work by pointing out that the most general extensions and generalizations of the Perron-Frobenius theory for nonnegative matrices, have been introduced, stated and proved. Our theory can be applied for the solution of linear systems derived from the discretisation of elliptic and parabolic partial differential equations, from integral equations, from Markov chains and from other applications. The introduced Perron-Frobenius splittings can also be used in connection with the multisplitting techniques in order to solve linear systems of the aforementioned applications on computers of parallel architecture.

References

[1] R. Bellman *Introduction to Matrix Analysis.* SIAM, Philadelphia, PA, 1995.

[2] A. Berman and R.J. Plemmons *Nonnegative Matrices in the Mathematical Sciences. Classics in Applied Mathematics.* SIAM, Philadelphia, PA, 1994.

[3] R. Bru, L. Elsner and M. Neumann, *Models of parallel chaotic iteration methods.* Linear Algebra Appl. 103 (1988), 175–192.

[4] J.J. Climent and C. Perea, *Some comparison theorems for weak nonnegative splittings of bounded operators.* Linear Algebra Appl. 275–276 (1998), 77–106.

[5] G. Csordas and R.S. Varga, *Comparison of regular splittings of matrices.* Numer. Math. 44 (1984), 23–35.

[6] L. Elsner, *Comparison of weak regular splittings and multisplitting methods.* Numer. Math. 56 (1989), 283–289.

[7] G. Frobenius, *Uber matrizen aus nicht negativen elementen.* S.-B. Preuss Acad. Wiss. Berlin (1912).

[8] I. Marek and D. B. Szyld, *Comparison theorems for weak splittings of bounded operators.* Numer. Math. 58 (1990), 387–397.

[9] V.A. Miller and M. Neumann, *A note on comparison theorems for nonnegative matricess.* Numer. Math. 47 (1985), 427–434.

[10] M. Neumann and R.J. Plemmons, *Convergent nonnegative matrices and iterative methods for consistent linear systems.* Numer. Math. 31 (1978), 265–279.

[11] M. Neumann and R.J. Plemmons, *Convergence of parallel multisplitting methods for $M-$matrices.* Linear Algebra Appl. 88 (1987), 559–574.

[12] E. Nteirmentzidis, *Classes of splittings of matrices and their convergence theory.* MSc Dissertation, Department of Mathematics, University of Ioannina, Greece, (2001), (in Greek).

[13] D.P. O'Leary and R.E. White, *Multi-splittings of matrices and parallel solutions of linear systems.* SIAM J. Algebraic Discrete Methods 6 (1985), 630–640.

[14] O. Perron, *Zur theorie der matrizen*, Math. Ann. 64 (1907), 248–263.

[15] P. Tarazaga, M Raydan and A. Hurman, *Perron-Frobenius theorem for matrices with some negative entries.* Linear Algebra Appl. 328 (2001), 57–68.

[16] R.S. Varga, *Matrix Iterative Analysis.* Prentice-Hall, Englewood Cliffs, NJ, 1962. (Also: 2nd Edition, Revised and Expanded, Springer, Berlin, 2000.)

[17] R.E. White, *Multisplitting with different weighting schemes.* SIAM J. Matrix Anal. Appl. 10 (1989), 481–493.

[18] Z. Woźnicki, *Two-sweep iterative methods for solving large linear systems and their application to the numerical solution of multi-group multi-dimensional neutron diffusion equation*, Doctoral Dissertation, Institute of Nuclear Research, Świerk k/Otwocka, Poland, (1973).

[19] Z. Woźnicki, *Nonnegative Splitting Theory*, Japan Journal of Industrial and Applied Mathematics 11 (1994), 289–342.

[20] D.M. Young, *Iterative Solution of Large Linear Systems.* Academic Press, New York, 1971.